

DISCRETE APPROXIMATIONS ON FUNCTIONAL CLASSES FOR THE INTEGRABLE NONLINEAR SCHRÖDINGER DYNAMICAL SYSTEM: A SYMPLECTIC FINITE-DIMENSIONAL REDUCTION APPROACH

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ABSTRACT. We investigate discretizations of the integrable discrete nonlinear Schrödinger dynamical system and related symplectic structures. We develop an effective scheme of invariant reducing the corresponding infinite system of ordinary differential equations to an equivalent finite system of ordinary differential equations with respect to the evolution parameter. We construct a finite set of recurrent algebraic regular relations allowing to generate solutions of the discrete nonlinear Schrödinger dynamical system and we discuss the related functional spaces of solutions. Finally, we discuss the Fourier transform approach to studying the solution set of the discrete nonlinear Schrödinger dynamical system and its functional-analytical aspects.

1. INTRODUCTION

With its origins going back several centuries, discrete analysis becomes now an increasingly central methodology for many mathematical problems related to discrete systems and algorithms, widely applied in modern science. Our theme, being related with studying integrable discretizations of nonlinear integrable dynamical systems and the limiting properties of their solution sets, is of deep interest in many branches of modern science and technology, especially in discrete mathematics, numerical analysis, statistics and probability theory as well as in electrical and electronic engineering. In fact, this topic belongs to a much more general realm of mathematics, namely to calculus, differential equations and differential geometry. Thereby, although the topic is discrete, our approach to treating this problem will be completely analytical.

In this work we will analyze the properties of discrete approximation for the nonlinear integrable Schrödinger (NLS) dynamical system on a functional manifold $\tilde{M} \subset L_2(\mathbb{R}; \mathbb{C}^2)$:

$$(1.1) \quad \left. \begin{aligned} \frac{d}{dt} \psi &= i\psi_{xx} - 2i\alpha\psi\psi^*, \\ \frac{d}{dt} \psi^* &= -i\psi_{xx}^* + 2i\alpha\psi^*\psi \end{aligned} \right\} := \tilde{K}[\psi, \psi^*],$$

where, by definition $(\psi, \psi^*)^\top \in \tilde{M}$, $\alpha \in \mathbb{R}$ is a constant, the subscript "x" means the partial derivative with respect to the independent variable $x \in \mathbb{R}$, $\tilde{K} : \tilde{M} \rightarrow T(\tilde{M})$ is the corresponding vector field on \tilde{M} and $t \in \mathbb{R}$ is the evolution parameter. The system (1.1) possesses a Lax type representation (see [20]) and is Hamiltonian

$$(1.2) \quad \frac{d}{dt}(\psi, \psi^*)^\top = -\tilde{\theta} \text{grad} \tilde{H}[\psi, \psi^*] = \tilde{K}[\psi, \psi^*]$$

with respect to the canonical Poisson structure $\tilde{\theta}$ and the Hamiltonian function \tilde{H} , where

$$(1.3) \quad \tilde{\theta} := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

is a non-degenerate mapping $\tilde{\theta} : T^*(\tilde{M}) \rightarrow T(\tilde{M})$ on the smooth functional manifold \tilde{M} , and

$$(1.4) \quad \tilde{H} := \frac{1}{2} \int_{\mathbb{R}} dx [\psi\psi_{xx}^* + \psi_{xx}\psi^* - 2\alpha(\psi^*\psi)^2],$$

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is a smooth mapping $\tilde{H} : \tilde{M} \rightarrow \mathbb{C}$. The corresponding symplectic structure [4, 5, 7, 9] for the Poissonian operator (1.3) is defined by

$$(1.5) \quad \begin{aligned} \tilde{\omega}^{(2)} &: = -\frac{i}{2} \int_{\mathbb{R}} dx [< (d\psi, d\psi^*)^\top, \wedge \tilde{\theta}^{-1}(d\psi, d\psi^*)^\top > = \\ &= -i \int_{\mathbb{R}} dx [d\psi^*(x) \wedge d\psi(x)], \end{aligned}$$

which is a non-degenerate and closed 2-form on the functional manifold \tilde{M} .

The simplest spatial discretizations of the dynamical system (1.1) look as the flows

$$(1.6) \quad \begin{aligned} \frac{d}{dt} \psi_n &= \frac{i}{h^2} (\psi_{n+1} - 2\psi_n + \psi_{n-1}) - 2i\alpha \psi_n \psi_n^*, \\ \frac{d}{dt} \psi_n^* &= -\frac{i}{h^2} (\psi_{n+1}^* - 2\psi_n^* + \psi_{n-1}^*) + 2i\alpha \psi_n^* \psi_n \end{aligned}$$

and

$$(1.7) \quad \left. \begin{aligned} \frac{d}{dt} \psi_n &= \frac{i}{h^2} (\psi_{n+1} - 2\psi_n + \psi_{n-1}) - i\alpha (\psi_{n+1} + \psi_{n-1}) \psi_n \psi_n^*, \\ \frac{d}{dt} \psi_n^* &= -\frac{i}{h^2} (\psi_{n+1}^* - 2\psi_n^* + \psi_{n-1}^*) + i\alpha (\psi_{n+1}^* + \psi_{n-1}^*) \psi_n \psi_n^*, \end{aligned} \right\} := K[\psi_n, \psi_n^*]$$

on some "discrete" submanifold M_h , where, by definition, $\{(\psi_n, \psi_n^*)^\top \in \mathbb{C}^2 : n \in \mathbb{Z}\} \subset M_h \subset l_2(\mathbb{Z}; \mathbb{C}^2)$ and $K : M_h \rightarrow T(M_h)$ is the corresponding vector field on M_h .

Definition 1.1. If for a function $(\psi, \psi^*)^\top \in W_2^2(\mathbb{R}; \mathbb{C}^2)$ there exists the point-wise limit $\lim_{h \rightarrow 0} (\psi_n, \psi_n^*)^\top = (\psi(x), \psi^*(x))^\top$, where the set of vectors $(\psi_n, \psi_n^*)^\top \in \mathbb{C}^2, n \in \mathbb{Z}$, solves the infinite system of equations (1.7), the set $\{(\psi_n, \psi_n^*)^\top \in \mathbb{C}^2 : n \in \mathbb{Z}\} \subset l_2(\mathbb{Z}; \mathbb{C}^2)$ will be called an approximate solution to the nonlinear Schrödinger dynamical system (1.1).

It is well known [1, 2] that the discretization scheme (1.7) conserves the Lax type integrability [20, 7, 9] and that the scheme (1.6) does not. The integrability of (1.7) can be easily enough checked by means of either the gradient-holonomic integrability algorithm [22, 21, 9] or the well known [18] symmetry approach. In particular, the discrete dynamical system (1.7) is a Hamiltonian one [4, 5, 7, 22] on the symplectic manifold $M_h \subset l_2(\mathbb{Z}; \mathbb{C}^2)$ with respect to the non-canonical symplectic structure

$$(1.8) \quad \omega_h^{(2)} = - \sum_{n \in \mathbb{Z}} \frac{ih}{2(1 - h^2 \alpha \psi_n^* \psi_n)} < (d\psi_n, d\psi_n^*)^\top, \wedge (d\psi_n, d\psi_n^*)^\top >$$

on M_h and looks as

$$(1.9) \quad \frac{d}{dt} (\psi_n, \psi_n^*)^\top = -\theta_n \text{grad} H[\psi_n, \psi_n^*] = K[\psi_n, \psi_n^*],$$

where the Hamiltonian function

$$(1.10) \quad H = \sum_{n \in \mathbb{Z}} \frac{1}{h} (\psi_n \psi_{n+1}^* + \psi_{n+1} \psi_n^* + \frac{2}{\alpha h^2} \ln |1 - \alpha h^2 \psi_n^* \psi_n|)$$

and the related Poissonian operator $\theta_n : T^*(M_h) \rightarrow T(M_h)$ equals

$$(1.11) \quad \theta_n := \begin{pmatrix} 0 & -ih^{-1}(1 - h^2 \alpha \psi_n^* \psi_n) \\ ih^{-1}(1 - h^2 \alpha \psi_n^* \psi_n) & 0 \end{pmatrix}.$$

Remark 1.2. For the symplectic structure (1.8) and, respectively, the Hamiltonian function (1.10) to be suitably defined on the manifold $M_h \subset l_2(\mathbb{Z}; \mathbb{C}^2)$ it is necessary to assume additionally that the finite stability condition $\lim_{N, M \rightarrow \infty} \left(\prod_{-N}^M (1 - \alpha h^2 \psi_n^* \psi_n) \right) \neq 0$ holds. The latter is simply

reduced as $h \rightarrow 0$ to the equivalent integral inequality $\alpha \leq \int_{\mathbb{R}} (x \psi^* \psi)^2 dx \left(\int_{\mathbb{R}} \psi^* \psi dx \right)^{-1}$, which will be assumed for further to be satisfied. respectively, the manifold $\tilde{M} \subset \tilde{W}_2^2(\mathbb{R}; \mathbb{C}^2)$, where $\tilde{W}_2^2(\mathbb{R}; \mathbb{C}^2) := W_2^2(\mathbb{R}; \mathbb{C}^2) \cap L_2^{(1)}(\mathbb{R}; \mathbb{C}^2)$ with the space $L_2^{(1)}(\mathbb{R}; \mathbb{C}^2) := \{(\psi, \psi^*)^\top \in L_2(\mathbb{R}; \mathbb{C}^2) : \int_{\mathbb{R}} x^2 (\psi^* \psi)^2 dx < \infty\}$.

The symplectic structure (1.8) is well defined on the manifold M_h and tends as $h \rightarrow 0$ to the symplectic structure (1.5) on \tilde{M} , and respectively the Hamiltonian function (1.10) tends to (1.4).

In this work we have investigated the structure of the solution set to the discrete nonlinear Schrödinger dynamical system (1.7) by means of a specially devised analytical approach for invariant reducing the infinite system of ordinary differential equations (1.7) to an equivalent finite one of ordinary differential equations with respect to the evolution parameter $t \in \mathbb{R}$. As a result, there was constructed a finite set of recurrent algebraic regular relationships, allowing to expand the obtained before finite set of solutions to any discrete order $n \in \mathbb{Z}$, which makes it possible to present a wide class of the approximate solutions to the nonlinear Schrödinger dynamical system (1.1). It is worthy here to stress that the problem of constructing an effective discretization scheme for the nonlinear Schrödinger dynamical system (1.1) and its generalizations proves to be important both for applications [3, 16, 24, 25] and for deeper understanding the nature of the related algebro-geometric and analytic structures responsible for their limiting stability and convergence properties. From these points of view we would like to mention work [17], where the standard discrete Lie-algebraic approach [7, 8] was recently applied to constructing a slightly different from (1.6) and (1.7) discretization of the nonlinear Schrödinger dynamical system (1.1). As the symplectic reduction method, devised in the present work for studying the solution sets of the discrete nonlinear Schrödinger dynamical system (1.7), is completely independent of a chosen discretization scheme, it would be reasonable and interesting to apply it to that of [17] and compare the corresponding results subject to their computational effectiveness.

2. A CLASS OF HAMILTONIAN DISCRETIZATIONS OF THE NLS DYNAMICAL SYSTEM

The discretizations (1.6) and (1.7) can be extended to a wide class of Hamiltonian systems, if to assume that the Poissonian structure is given by the local expression

$$(2.1) \quad \theta_n = \begin{pmatrix} 0 & -i\nu_n(g_n - \tilde{h}_n^2 \alpha \psi_n^* \psi_n) \\ i\nu_n(g_n - \tilde{h}_n^2 \alpha \psi_n^* \psi_n) & 0 \end{pmatrix},$$

generalizing (1.11), and the Hamiltonian function is chosen in the form

$$(2.2) \quad H = \sum_{n \in \mathbb{Z}} h_n \left(a_n \psi_n \psi_{n+1}^* + b_n \psi_n \psi_n^* + c_n \psi_n \psi_{n-1}^* + \frac{2d_n}{\alpha} \ln |g_n - \alpha \tilde{h}_n^2 \psi_n \psi_n^*| \right),$$

where $h_n, \tilde{h}_n, \nu_n, a_n, b_n, c_n, d_n$ and $g_n \in \mathbb{R}_+, n \in \mathbb{Z}$, are some parameters. The reality condition, imposed on the Hamiltonian function (2.2), yields the relationships

$$(2.3) \quad c_n h_n = a_{n-1}^* h_{n-1}, \quad b_n^* = b_n, \quad d_n^* = d_n,$$

which should be satisfied for all $n \in \mathbb{Z}$. As a result, there is obtained the corresponding generalized discrete nonlinear Schrödinger dynamical system $\frac{d}{dt}(\psi_n, \psi_n^*)^\top := -\theta_n \text{grad } H[\psi_n, \psi_n^*]$, $n \in \mathbb{Z}$, equivalent to the infinite set of ordinary differential equations

$$(2.4) \quad \begin{aligned} \frac{d}{dt} \psi_n &= i\nu_n \left(h_{n+1} c_{n+1} g_n \psi_{n+1} + (b_n g_n h_n - 2\tilde{h}_n^2 h_n d_n) \psi_n + h_{n-1} a_{n-1} g_n \psi_{n-1} \right) - \\ &\quad - i\alpha \nu_n \tilde{h}_n^2 (h_{n+1} c_{n+1} \psi_{n+1} + h_n b_n \psi_n + h_{n-1} a_{n-1} \psi_{n-1}) \psi_n \psi_n^* \\ \frac{d}{dt} \psi_n^* &= -i\nu_n \left(h_n a_n g_n \psi_{n+1}^* + (b_n g_n h_n - 2\tilde{h}_n^2 h_n d_n) \psi_n^* + h_n c_n g_n \psi_{n-1}^* \right) + \\ &\quad + i\alpha \nu_n \tilde{h}_n^2 (h_n a_n \psi_{n+1}^* + h_n b_n \psi_n^* + h_n c_n \psi_{n-1}^*) \psi_n \psi_n^* \end{aligned}$$

for all $n \in \mathbb{Z}$. In the completely autonomous case, when $h_n = h, \tilde{h}_n = \tilde{h}, \nu_n = \nu, a_n = a, b_n = b, c_n = c, d_n = d$ and $g_n = g \in \mathbb{R}_+$ for all $n \in \mathbb{Z}$, the Poissonian structure (2.1) becomes

$$(2.5) \quad \theta_n = \begin{pmatrix} 0 & -i\nu(g - \tilde{h}^2 \alpha \psi_n^* \psi_n) \\ i\nu(g - \tilde{h}^2 \alpha \psi_n^* \psi_n) & 0 \end{pmatrix}$$

and the Hamiltonian function (2.2) becomes

$$(2.6) \quad H = \sum_{n \in \mathbb{Z}} h \left(a \psi_n \psi_{n+1}^* + b \psi_n \psi_n^* + c \psi_n \psi_{n-1}^* + \frac{2d}{\alpha} \ln |g - \alpha \tilde{h}^2 \psi_n \psi_n^*| \right).$$

The corresponding reality condition for (2.6) reads as

$$(2.7) \quad c = a^*, \quad b^* = b, \quad d^* = d,$$

and the related discrete nonlinear Schrödinger dynamical systems reads as a set of the equations

$$(2.8) \quad \begin{aligned} \frac{d}{dt} \psi_n &= i\nu h \left(c g \psi_{n+1} + (b g - 2\tilde{h}^2 d) \psi_n + a g \psi_{n-1} \right) - i\alpha \nu h \tilde{h}^2 (c \psi_{n+1} + b \psi_n + a \psi_{n-1}) \psi_n \psi_n^*, \\ \frac{d}{dt} \psi_n^* &= -i\nu h \left(a g \psi_{n+1}^* + (b g - 2\tilde{h}^2 d) \psi_n^* + c g \psi_{n-1}^* \right) + i\alpha \nu h \tilde{h}^2 (a \psi_{n+1}^* + b \psi_n^* + c \psi_{n-1}^*) \psi_n \psi_n^*, \end{aligned}$$

for all $n \in \mathbb{Z}$. If now to make in (2.4) the substitutions

$$(2.9) \quad \begin{aligned} \nu_n &= \frac{1}{h_n}, \quad g_n = 1, \quad \tilde{h}_n = h_n, \quad a_n = \frac{1}{h_n^2}, \\ b_n &= 0, \quad c_n = \frac{1}{h_n h_{n-1}}, \quad d_n = \frac{1}{h_n^4}, \end{aligned}$$

one obtains the discrete nonlinear Schrödinger dynamical system

$$(2.10) \quad \left. \begin{aligned} \frac{d}{dt} \psi_n &= \frac{i}{h_n^2} (\psi_{n+1} - 2\psi_n + h_n h_{n-1}^{-1} \psi_{n-1}) - \\ &\quad - i\alpha (\psi_{n+1} + h_n h_{n-1}^{-1} \psi_{n-1}) \psi_n^* \psi_n, \\ \frac{d}{dt} \psi_n^* &= -\frac{i}{h_n^2} (\psi_{n+1}^* - 2\psi_n^* + h_n h_{n-1}^{-1} \psi_{n-1}^*) + \\ &\quad + i\alpha (\psi_{n+1}^* + h_n h_{n-1}^{-1} \psi_{n-1}^*) \psi_n^* \psi_n, \end{aligned} \right\} := K_n^{(g)}[\psi_n, \psi_n^*],$$

whose Hamiltonian function equals

$$(2.11) \quad H^{(g)} = \sum_{n \in \mathbb{Z}} h_n^{-1} (\psi_n \psi_{n+1}^* + \psi_{n+1} \psi_n^* + \frac{2}{\alpha h_n^2} \ln |1 - \alpha h_n^2 \psi_n^* \psi_n|).$$

Another substitution, taken in the form

$$(2.12) \quad c = a \neq 0, \quad \nu h g a = \frac{1}{h^2}, \quad (b g - 2\tilde{h}^2 d) \nu h = -\frac{2}{h^2}, \quad \nu h \tilde{h}^2 (a + b + c) = 2,$$

is also suitable in the limit $h \rightarrow 0$ for discretization the nonlinear Schrödinger dynamical system (1.7). The corresponding discrete nonlinear Schrödinger dynamics takes the form

$$(2.13) \quad \begin{aligned} \frac{d}{dt} \psi_n &= \frac{i}{h^2} (\psi_{n+1} - 2\psi_n + \psi_{n-1}) - \frac{2i\alpha}{2+\mu} (\psi_{n+1} + \psi_{n-1} + \mu \psi_n) \psi_n \psi_n^*, \\ \frac{d}{dt} \psi_n^* &= -\frac{i}{h^2} (\psi_{n+1}^* - 2\psi_n^* + \psi_{n-1}^*) + \frac{2i\alpha}{2+\mu} (\psi_{n+1}^* + \psi_{n-1}^* + \mu \psi_n^*) \psi_n \psi_n^* \end{aligned}$$

for all $n \in \mathbb{Z}$, where $\mu = b/a \in \mathbb{R}_+$. Thus we obtained a one-parameter family of Hamiltonian discretizations of the NLS equation. The set of relationships (2.12) admits a lot of reductions, for instance, one can take

$$(2.14) \quad \nu = 1, \quad g = 1, \quad a = \frac{1}{h^3}, \quad d = \left(\frac{\mu + 2}{2} \right)^2 \frac{1}{h^5}, \quad \frac{\tilde{h}^2}{h^2} = \frac{2}{2 + \mu},$$

not changing the infinite set of equations (2.13).

All of the constructed above discretizations of the nonlinear Schrödinger dynamical system (1.1) on the functional manifold \tilde{M} can be considered as either better or worse from the computational point of view. If some of the discretization allows, except the Hamiltonian function, some extra conservation laws, it can be naturally considered as a much more suitable for numerical analysis case, allowing both to control the stability of the solution convergence, as a parameter $\mathbb{R}_+ \ni h \rightarrow 0$, and to make an invariant solution space reduction to a lower effective dimension of the related solution set.

It is worthy to observe here that the functional structure of the discretization (1.7) strongly depends both on the manifold M and on the convergent as $h \rightarrow 0$ form of the Hamiltonian function (1.11). In particular, the existence of the limit

$$(2.15) \quad \tilde{H} := \lim_{h \rightarrow 0} \sum_{n \in \mathbb{Z}} \frac{1}{h} (\psi_n \psi_{n+1}^* + \psi_{n+1} \psi_n^* + \frac{2}{\alpha h^2} \ln |1 - \alpha h^2 \psi_n^* \psi_n|),$$

coinciding with the expression (1.4), imposes a strong constraint on the functional space $\tilde{M} \subset L_2(\mathbb{R}; \mathbb{C}^2)$, namely, a vector-function $(\psi, \psi^*)^\top \in W_2^2(\mathbb{R}; \mathbb{C}^2) \subset L_2(\mathbb{R}; \mathbb{C}^2)$, thereby fixing a suitable

functional class [6] for which the discretization conserves its physical Hamiltonian system sense. Respectively, the limiting for (2.15) symplectic structure

$$\begin{aligned}
 (2.16) \quad \tilde{\omega}^{(2)} &: = -\lim_{h \rightarrow 0} \sum_{n \in \mathbb{Z}} \frac{i}{2} \langle (d\psi_n, d\psi_n^*)^\top, \wedge \theta_n^{-1} (d\psi_n, d\psi_n^*)^\top \rangle = \\
 &= -\lim_{h \rightarrow 0} i \sum_{n \in \mathbb{Z}} h (1 - \alpha h^2 \psi_n^* \psi_n)^{-1} d\psi_n^* \wedge d\psi_n = -i \int_{\mathbb{R}} dx [d\psi^*(x) \wedge d\psi(x)]
 \end{aligned}$$

on the manifold \tilde{M} coincides exactly with the canonical symplectic structure (1.5) for the dynamical system (1.2).

If now to assume that a vector function $(\psi, \psi^*)^\top \in W_2^1(\mathbb{R}; \mathbb{C}^2) \subset L_2(\mathbb{R}; \mathbb{C}^2)$, the Hamiltonian function (1.11) can be taken only as

$$(2.17) \quad H^{(s)} = \sum_{n \in \mathbb{Z}} (\psi_n \psi_{n+1}^* + \psi_{n+1} \psi_n^* + \frac{2}{\alpha h^2} \ln |1 - \alpha h^2 \psi_n^* \psi_n|),$$

and the corresponding Poissonian structure as

$$(2.18) \quad \theta_n^{(s)} := \begin{pmatrix} 0 & i h^{-2} (h^2 \alpha \psi_n^* \psi - 1) \\ i h^{-2} (1 - h^2 \alpha \psi_n^* \psi) & 0 \end{pmatrix}$$

The limiting for (2.17) Hamiltonian function

$$(2.19) \quad \tilde{H}^{(s)} := \lim_{h \rightarrow 0} H^{(s)} = \int_{\mathbb{R}} dx (\psi \psi_x^* + \psi_x \psi^*) = 0$$

becomes trivial and, simultaneously, the limiting for (2.18) symplectic structure

$$\begin{aligned}
 (2.20) \quad \tilde{\omega}_{(s)}^{(2)} &: = \lim_{h \rightarrow 0} \sum_{n \in \mathbb{Z}} \frac{i}{2} \langle (d\psi_n, d\psi_n^*)^\top, \wedge \theta_n^{(s), -1} (d\psi_n, d\psi_n^*)^\top \rangle = \\
 &= \lim_{h \rightarrow 0} i \sum_{n \in \mathbb{Z}} h^2 (1 - \alpha h^2 \psi_n^* \psi_n)^{-1} d\psi_n^* \wedge d\psi_n = 0
 \end{aligned}$$

becomes trivial too. Thus, the functional space $W_2^1(\mathbb{R}; \mathbb{C}^2) \subset L_2(\mathbb{R}; \mathbb{C}^2)$ is not suitable for the discretization (1.7) of the nonlinear integrable Schrödinger dynamical system (1.1).

It is important here to stress that the discretization parameter $h \in \mathbb{R}_+$ can be taken as depending on the node $n \in \mathbb{Z} : h \rightarrow h_n \in \mathbb{R}_+$, which satisfies the condition $\sup_{n \in \mathbb{Z}} h_n \leq \varepsilon$, where the condition $\varepsilon \rightarrow 0$ should be later imposed. For instance, one can replace the dynamical system (1.7) by (2.10), the Poissonian structure (1.8) by

$$(2.21) \quad \theta_n^{(g)} := \begin{pmatrix} 0 & i h_n^{-1} (h_n^2 \alpha \psi_n^* \psi - 1) \\ i h_n^{-1} (1 - h_n^2 \alpha \psi_n^* \psi) & 0 \end{pmatrix}$$

and, respectively, the Hamiltonian function (1.11) becomes exactly (2.11).

It is easy to check that the modified discrete dynamical system (2.10) can be equivalently rewritten as

$$(2.22) \quad \frac{d}{dt} (\psi_n, \psi_n^*)^\top = -\theta_n^{(g)} \text{grad} H^{(g)} [\psi_n, \psi_n^*]$$

for all $n \in \mathbb{Z}$, meaning, in particular, that the Hamiltonian function (2.11) is conservative. The latter follows from the fact that the skewsymmetric operator (2.21) is Poissonian on the discretized manifold M_h . Moreover, if to impose the constraint that uniformly in $n \in \mathbb{Z}$ the limit $\lim_{\varepsilon \rightarrow 0} (h_n h_{n-1}^{-1}) = 1$, the dynamical system (2.10) reduces to (1.1) and the corresponding limiting symplectic structure

$$\begin{aligned}
 (2.23) \quad \tilde{\omega}_{(g)}^{(2)} &: = -\lim_{\varepsilon \rightarrow 0} \sum_{n \in \mathbb{Z}} \frac{i}{2} \langle (d\psi_n, d\psi_n^*)^\top, \wedge \theta_n^{(g), -1} (d\psi_n, d\psi_n^*)^\top \rangle = \\
 &= -\lim_{\varepsilon \rightarrow 0} i \sum_{n \in \mathbb{Z}} h_n (1 - \alpha h_n^2 \psi_n^* \psi_n)^{-1} d\psi_n^* \wedge d\psi_n = \\
 &= -i \int_{\mathbb{R}} dx [d\psi^*(x) \wedge d\psi(x)],
 \end{aligned}$$

coincides exactly with the symplectic structure (2.16).

Remark 2.1. It is, by now, a not solved, but interesting, problem whether the modified discrete Hamiltonian dynamical system (2.10) sustains to be Lax type integrable. It is left for studying in a separate work.

3. CONSERVATION LAWS FOR THE INTEGRABLE DISCRETE NLS SYSTEM

Taking into account that the discrete dynamical system (1.7) is well posed in the space $M_h := w_{h,2}^2(\mathbb{Z}; \mathbb{C}^2) \subset l_2(\mathbb{Z}; \mathbb{C}^2)$, suitably approximating the Sobolev space of functions $W_2^2(\mathbb{R}; \mathbb{C}^2)$, we can go further and to approximate the space $w_{h,2}^2(\mathbb{Z}; \mathbb{C}^2)$ by means of an infinite hierarchy of strictly invariant finite dimensional subspaces $M_h^{(N)} \simeq \bar{w}_{h,2}^2(\mathbb{Z}_{(N)}; \mathbb{C}^2)$, $N \in \mathbb{Z}_+$. In particular, as it was before shown both in [1, 2] by means of the inverse scattering transform method [1, 20] and in [10, 22, 21] by means of the gradient-holonomic approach [23], the discrete nonlinear Schrödinger dynamical system (1.7) possesses on the manifold M_h an infinite hierarchy of the functionally independent conservation laws:

$$(3.1) \quad \begin{aligned} \bar{\gamma}_0 &= \frac{1}{\alpha h^3} \sum_{n \in \mathbb{Z}} \ln |1 - \alpha h^2 \psi_n^* \psi_n|, & \gamma_0 &= \sum_{n \in \mathbb{Z}_+} \sigma_n^{(0)}, \\ \gamma_1 &= \sum_{n \in \mathbb{Z}} (\sigma_n^{(1)} + \frac{1}{2} \sigma_n^{(0)} \sigma_n^{(0)}), \\ \gamma_2 &= \sum_{n \in \mathbb{Z}} (\sigma_n^{(2)} + \frac{1}{3} \sigma_n^{(0)} \sigma_n^{(0)} \sigma_n^{(0)} + \sigma_n^{(0)} \sigma_n^{(1)}), \quad \dots, \end{aligned}$$

where the quantities $\sigma_n^{(j)}$, $n \in \mathbb{Z}$, $j \in \mathbb{Z}_+$, are defined as follows:

$$(3.2) \quad \begin{aligned} \sigma_n^{(0)} &= -\frac{1}{\alpha h^2} (\psi_n^* \psi_{n-1} + \psi_{n-1}^* \psi_{n-2}), \\ \sigma_n^{(1)} &= i \frac{d}{dt} \sigma_{n-1}^{(0)} + (1 - \alpha h^2 \psi_{n-1}^* \psi_{n-1}) (1 - \alpha h^2 \psi_{n-2}^* \psi_{n-2}) + \beta \frac{\alpha}{h^2} \psi_{n-1}^* (\psi_n + \psi_{n-1}), \quad \dots, \end{aligned}$$

and $\beta \in \mathbb{R}$ is an arbitrary constant parameter. As a result of (3.2) one finds the following infinite hierarchy of smooth conservation laws:

$$(3.3) \quad \begin{aligned} \bar{H}_0 &= \sum_{n \in \mathbb{Z}} \ln |1 - \alpha h^2 \psi_n^* \psi_n|, \\ H_0 &= \sum_{n \in \mathbb{Z}} \psi_n^* \psi_{n+1}, & H_0^* &= \sum_{n \in \mathbb{Z}} \psi_n \psi_{n+1}^*, \\ H_1 &= \sum_{n \in \mathbb{Z}} (\frac{1}{2} \psi_n^2 \psi_{n-1}^{*,2} + \psi_n \psi_{n+1} \psi_{n-1}^* \psi_n^* - \frac{\psi_n \psi_{n-2}^*}{\alpha h^2}), \\ H_1^* &= \sum_{n \in \mathbb{Z}} (\frac{1}{2} \psi_{n-1}^2 \psi_n^{*,2} + \psi_{n-1} \psi_n \psi_{n+1}^* \psi_n^* - \frac{\psi_{n-2} \psi_n^*}{\alpha h^2}), \\ H_2 &= \sum_{n \in \mathbb{Z}} [\frac{1}{3} \psi_n^3 \psi_{n-1}^{*,3} + \psi_n \psi_{n+1} \psi_{n-1}^* \psi_n^* (\psi_n \psi_{n-1}^* + \psi_{n+1} \psi_n^* + \\ &\quad + \psi_{n+2} \psi_{n+1}^*) - \frac{\psi_n \psi_{n-1}^*}{\alpha h^2} (\psi_n \psi_{n-2}^* + \psi_{n+1} \psi_{n-1}^*) - \\ &\quad - \frac{\psi_n \psi_n^*}{\alpha h^2} (\psi_{n+1} \psi_{n-2}^* + \psi_{n+2} \psi_{n-1}^*) + \frac{\psi_n \psi_{n-3}^*}{\alpha^2 h^4}], \\ H_2^* &= \sum_{n \in \mathbb{Z}} [\frac{1}{3} \psi_n^{*,3} \psi_{n-1}^3 + \psi_n^* \psi_{n+1}^* \psi_{n-1} \psi_n (\psi_n^* \psi_{n-1} + \psi_{n+1}^* \psi_n + \\ &\quad + \psi_{n+2}^* \psi_{n+1}) - \frac{\psi_n^* \psi_{n-1}}{\alpha h^2} (\psi_n^* \psi_{n-2} + \psi_{n+1}^* \psi_{n-1}) - \\ &\quad - \frac{\psi_n \psi_n^*}{\alpha h^2} (\psi_{n+1}^* \psi_{n-2} + \psi_{n+2}^* \psi_{n-1}) + \frac{\psi_n^* \psi_{n-3}}{\alpha^2 h^4}], \end{aligned}$$

and so on.

Taking into account the functional structure of the equations (1.6) or (1.7), one can define the space $\mathcal{D}(M_h)$ of smooth functions $\gamma : M_h \rightarrow \mathbb{C}$ on M_h as that invariant with respect to the phase

transformation $\mathbb{C}^2 \ni (\psi_n, \psi_n^*) \rightarrow (e^\alpha \psi_n, e^{-\alpha} \psi_n^*) \in \mathbb{C}^2$ for any $n \in \mathbb{Z}$ and $\alpha \in \mathbb{C}$. Equivalently, a function $\gamma \in \mathcal{D}(M_h)$ iff the following condition

$$(3.4) \quad \sum_{n \in \mathbb{Z}} \langle \text{grad} \gamma[\psi_n, \psi_n^*], (\psi_n, -\psi_n^*)^\top \rangle = 0$$

holds on M_h . Note that conserved quantities (3.3) belong to $\mathcal{D}(M_h)$.

The conservation law $\bar{H}_0 \in \mathcal{D}(M_h)$ is a Casimir function for the Poissonian structure (1.8) on the manifold M_h , that is for any $\gamma \in \mathcal{D}(M_h)$ the Poisson bracket

$$(3.5) \quad \begin{aligned} \{\gamma, \bar{H}_0\} &: = \sum_{n \in \mathbb{Z}} \langle \text{grad} \gamma[\psi_n, \psi_n^*], \theta_n \text{grad} \bar{H}_0[\psi_n, \psi_n^*] \rangle = \\ &= i\alpha h \sum_{n \in \mathbb{Z}} \langle \text{grad} \gamma[\psi_n, \psi_n^*], (\psi_n, -\psi_n^*) \rangle = 0, \end{aligned}$$

owing to the condition (3.4). The Hamiltonian function (1.11) is obtained from the first three invariants of (3.3) as

$$(3.6) \quad H = \frac{2}{\alpha h^3} \bar{H}_0 + \frac{1}{h} (H_0 + H_0^*).$$

Remark 3.1. Similarly to the limiting condition (2.15), the same limiting expression one obtains from the discrete invariant function

$$(3.7) \quad H^{(w)} = \frac{1}{2\alpha h^3} \bar{H}_0 - \frac{\alpha h}{4} (H_1 + H_1^*),$$

that is

$$(3.8) \quad \lim_{h \rightarrow 0} H^{(w)} = \tilde{H} := \frac{1}{2} \int_{\mathbb{R}} dx [\psi \psi_{xx}^* + \psi_{xx} \psi^* - 2\alpha (\psi^* \psi)^2].$$

Based on these results, one can apply to the discrete dynamical system (1.7) the Bogoyavlensky-Novikov type reduction scheme, devised before in [20, 21] and obtain a completely Liouville integrable finite dimensional dynamical system on the manifold $M_h^{(N)}$. Namely, we consider the critical submanifold $M_h^{(N)} \subset M_h$ of the following real-valued action functional

$$(3.9) \quad \mathcal{L}_h^{(N)} := \sum_{n \in \mathbb{Z}} \mathcal{L}_h^{(N)}[\psi_n, \psi_n^*] = \bar{c}_0(h) \bar{H}_0 + \sum_{j=0}^N c_j(h) (H_j + H_j^*),$$

where, by definition, $\bar{c}_0, c_j : \mathbb{R}_+ \rightarrow \mathbb{R}$, $j = \overline{0, N}$, are suitably defined functions for arbitrary but fixed $N \in \mathbb{Z}_+$, and

$$(3.10) \quad M_h^{(N)} := \left\{ (\psi, \psi^*)^\top \in M_h : \text{grad} \mathcal{L}_h^{(N)}[\psi_n, \psi_n^*] = 0, n \in \mathbb{Z} \right\}.$$

As one can easily show, the submanifold $M_h^{(N)} \subset M_h$ is finite-dimensional and for any $N \in \mathbb{Z}_+$ is invariant with respect to the vector field $K : M_h \rightarrow T(M_h)$. This property makes it possible to reduce it on the submanifold $M_h^{(N)} \subset M_h$ and to obtain a resulting finite-dimensional system of ordinary differential equations on $M_h^{(N)}$, whose solution manifold coincides with an subspace of exact solutions to the initial dynamical system (1.7). The latter proves to be canonically Hamiltonian on the manifold $M_h^{(N)}$ and, moreover, completely Liouville-Arnold integrable. If the mappings $\bar{c}_0, c_j : \mathbb{R}_+ \rightarrow \mathbb{R}$, $j = \overline{0, N}$, are chosen in such a way that the flow (1.7), invariantly reduced on the finite dimensional submanifold $M_h^{(N)} \subset M_h$, is nonsingular as $h \rightarrow 0$ and complete, then the corresponding solutions to the discrete dynamical system (1.7) will respectively approach those to the nonlinear integrable Schrödinger dynamical system (1.1).

Below we will proceed to realizing this scheme for the most simple cases $N = 1$ and $N = 2$. Another way of analyzing the discrete dynamical system (1.7), being interesting enough, consists in applying the approaches recently devised in [12, 19] and based on the long-time behavior of the chosen discretization subject to a fixed Hamiltonian function structure.

4. THE FINITE DIMENSIONAL REDUCTION SCHEME: THE CASE $N = 1$

Consider the following non degenerate action functional

$$\begin{aligned}
 \mathcal{L}_h^{(1)} = & \sum_{n \in \mathbb{Z}} \bar{c}_0(h) \ln |1 - \alpha h^2 \psi_n^* \psi_n| + \sum_{n \in \mathbb{Z}} c_0(h) (\psi_n^* \psi_{n+1} + \psi_n \psi_{n+1}^*) + \\
 (4.1) \quad & + \sum_{n \in \mathbb{Z}} c_1(h) \left(\frac{1}{2} \psi_n^2 \psi_{n-1}^{*,2} + \psi_n \psi_{n+1} \psi_{n-1}^* \psi_n^* - \frac{\psi_n \psi_{n-2}^*}{\alpha h^2} + \right. \\
 & \left. + \frac{1}{2} \psi_n^2 \psi_{n+1}^{*,2} + \psi_n \psi_{n+1} \psi_{n+1}^* \psi_{n+2}^* - \frac{\psi_{n-1} \psi_{n+1}^*}{\alpha h^2} \right)
 \end{aligned}$$

with mappings $\bar{c}_0, c_j : \mathbb{R}_+ \rightarrow \mathbb{R}$, $j = \overline{0, 1}$, taken as

$$(4.2) \quad \bar{c}_0(h) = \frac{4\xi + 1}{2\alpha h^3}, \quad c_0(h) = \frac{\xi}{h}, \quad c_1(h) = \frac{\alpha h}{4},$$

and being easily determined for any $\xi \in \mathbb{R}$ from the condition for existence of a limit as $h \rightarrow 0$:

$$(4.3) \quad \tilde{\mathcal{L}}^{(1)} := \lim_{h \rightarrow 0} \mathcal{L}_h^{(1)}.$$

The corresponding invariant critical submanifold

$$(4.4) \quad M_h^{(1)} := \left\{ (\psi, \psi^*)^\top \in M_h : \text{grad} \mathcal{L}_h^{(1)}[\psi_n, \psi_n^*] = 0, \quad n \in \mathbb{Z} \right\}$$

is equivalent to the following system of discrete up-recurrent relationships with respect to indices $n \in \mathbb{Z}$:

$$\begin{aligned}
 \psi_{n+2} &= -\frac{-\bar{c}_0(h)/c_1(h)\psi_n}{\left(\frac{1}{\alpha h^2} - \psi_{n+1} \psi_{n+1}^*\right)\left(\frac{1}{\alpha h^2} - \psi_n \psi_n^*\right)} + \\
 &+ \frac{2\psi_{n-1}c_0(h)/c_1(h) + \psi_n(\psi_{n+1}\psi_{n-1}^* + \psi_{n-1}\psi_{n+1}^*)}{\left(\frac{1}{\alpha h^2} - \psi_{n+1} \psi_{n+1}^*\right)} + \\
 &+ \frac{(\psi_{n+1}^2 + \psi_{n-1}^2)\psi_n^* - \psi_{n-2}\left(\frac{1}{\alpha h^2} - \psi_{n-1} \psi_{n-1}^*\right)}{\left(\frac{1}{\alpha h^2} - \psi_{n+1} \psi_{n+1}^*\right)} := \\
 &= \Phi_+(\psi_{n+1}, \psi_{n+1}^*; \psi_n, \psi_n^*; \psi_{n-1}, \psi_{n-1}^*), \\
 (4.5) \quad \psi_{n+2}^* &= -\frac{-\bar{c}_0(h)/c_1(h)\psi_n^*}{\left(\frac{1}{\alpha h^2} - \psi_{n+1} \psi_{n+1}^*\right)\left(\frac{1}{\alpha h^2} - \psi_n \psi_n^*\right)} + \\
 &+ \frac{2\psi_{n-1}^*c_0(h)/c_1(h) + \psi_n^*(\psi_{n+1}\psi_{n-1}^* + \psi_{n-1}\psi_{n+1}^*)}{\left(\frac{1}{\alpha h^2} - \psi_{n+1} \psi_{n+1}^*\right)} + \\
 &+ \frac{(\psi_{n+1}^{*,2} + \psi_{n-1}^{*,2})\psi_n - \psi_{n-2}^*\left(\frac{1}{\alpha h^2} - \psi_{n-1} \psi_{n-1}^*\right)}{\left(\frac{1}{\alpha h^2} - \psi_{n+1} \psi_{n+1}^*\right)} := \\
 &= \Phi_+(\psi_{n+1}, \psi_{n+1}^*; \psi_n, \psi_n^*; \psi_{n-1}, \psi_{n-1}^*).
 \end{aligned}$$

The latter can be also rewritten as the system of down-recurrent mappings

$$\begin{aligned}
 \psi_{n-2} &= -\frac{-\bar{c}_0(h)/c_1(h)\psi_n}{\left(\frac{1}{\alpha h^2} - \psi_{n-1} \psi_{n-1}^*\right)\left(\frac{1}{\alpha h^2} - \psi_n \psi_n^*\right)} + \\
 (4.6) \quad &+ \frac{2\psi_{n-1}c_0(h)/c_1(h) + \psi_n(\psi_{n+1}\psi_{n-1}^* + \psi_{n-1}\psi_{n+1}^*)}{\left(\frac{1}{\alpha h^2} - \psi_{n-1} \psi_{n-1}^*\right)} + \\
 &+ \frac{(\psi_{n+1}^2 + \psi_{n-1}^2)\psi_n^* - \psi_{n+2}\left(\frac{1}{\alpha h^2} - \psi_{n+1} \psi_{n+1}^*\right)}{\left(\frac{1}{\alpha h^2} - \psi_{n-1} \psi_{n-1}^*\right)} := \\
 &= \Phi_-(\psi_{n-1}, \psi_{n-1}^*; \psi_n, \psi_n^*; \psi_{n+1}, \psi_{n+1}^*),
 \end{aligned}$$

$$\begin{aligned}
\psi_{n-2}^* &= -\frac{-\bar{c}_0(h)/c_1(h)\psi_n^*}{(\frac{1}{\alpha h^2} - \psi_{n-1}\psi_{n-1}^*)(\frac{1}{\alpha h^2} - \psi_n\psi_n^*)} + \\
&+ \frac{2\psi_{n-1}^*c_0(h)/c_1(h) + \psi_n^*(\psi_{n+1}\psi_{n-1}^* + \psi_{n-1}\psi_{n+1}^*)}{(\frac{1}{\alpha h^2} - \psi_{n-1}\psi_{n-1}^*)} + \\
&+ \frac{(\psi_{n+1}^{*,2} + \psi_{n-1}^{*,2})\psi_n - \psi_{n+2}^*(\frac{1}{\alpha h^2} - \psi_{n+1}\psi_{n+1}^*)}{(\frac{1}{\alpha h^2} - \psi_{n-1}\psi_{n-1}^*)} := \\
&= \Phi_-^*(\psi_{n-1}, \psi_{n-1}^*; \psi_n, \psi_n^*; \psi_{n+1}, \psi_{n+1}^*),
\end{aligned}$$

which also hold for all $n \in \mathbb{Z}$. The relationships (4.5) (or, the same, relationships (4.6)) mean that the whole submanifold $M_h^{(1)} \subset M_h$ is retrieved by means of the initial values $(\bar{\psi}_{-1}, \bar{\psi}_{-1}^*; \bar{\psi}_0, \bar{\psi}_0^*; \bar{\psi}_1, \bar{\psi}_1^*; \bar{\psi}_2, \bar{\psi}_2^*)^\top \in M_h^{(1)} \simeq \mathbb{C}^8$. Thereby, the submanifold $M_h^{(1)} \subset M_h^8$ is naturally diffeomorphic to the finite dimensional complex space M_h^8 . Taking into account the canonical symplecticity [22, 21] of the submanifold $M_h^{(1)} \simeq M_h^8$ and its invariance with respect to the vector field (1.7) one can easily reduce it on this submanifold $M_h^{(1)} \simeq M_h^8$ and obtain the following equivalent finite dimensional flow on the manifold M_h^8 :

$$\begin{aligned}
\frac{d}{dt}\psi_2 &= \frac{i}{h^2}[\Phi_+(\psi_2, \psi_2^*; \psi_1, \psi_1^*; \psi_0, \psi_0^*) - 2\psi_2 + \psi_1] - \\
&- i\alpha[\Phi_+(\psi_2, \psi_2^*; \psi_1, \psi_1^*; \psi_0, \psi_0^*) + \psi_1]\psi_2\psi_2^*, \\
\frac{d}{dt}\psi_1 &= \frac{i}{h^2}[\psi_2 - 2\psi_1 + \psi_0] - i\alpha(\psi_2 + \psi_0)\psi_1\psi_1^*, \\
\frac{d}{dt}\psi_0 &= \frac{i}{h^2}[\psi_1 - 2\psi_0 + \psi_{-1}] - i\alpha(\psi_1 + \psi_{-1})\psi_0\psi_0^*, \\
\frac{d}{dt}\psi_{-1} &= \frac{i}{h^2}[\psi_0 - 2\psi_{-1} + \Phi_-(\psi_{-1}, \psi_{-1}^*; \psi_0, \psi_0^*; \psi_1, \psi_1^*)] - \\
&- i\alpha[\psi_0 + \Phi_-(\psi_{-1}, \psi_{-1}^*; \psi_0, \psi_0^*; \psi_1, \psi_1^*)]\psi_{-1}\psi_{-1}^*,
\end{aligned}
\tag{4.7}$$

and

$$\begin{aligned}
\frac{d}{dt}\psi_{-1}^* &= -\frac{i}{h^2}[\psi_0^* - 2\psi_{-1}^* + \Phi_-^*(\psi_{-1}, \psi_{-1}^*; \psi_0, \psi_0^*; \psi_1, \psi_1^*)] + \\
&+ i\alpha[\psi_0^* + \Phi_-^*(\psi_{-1}, \psi_{-1}^*; \psi_0, \psi_0^*; \psi_1, \psi_1^*)]\psi_{-1}\psi_{-1}^*, \\
\frac{d}{dt}\psi_0^* &= -\frac{i}{h^2}[\psi_1^* - 2\psi_0^* + \psi_{-1}^*] + i\alpha(\psi_1^* + \psi_{-1}^*)\psi_0\psi_0^*, \\
\frac{d}{dt}\psi_1^* &= -\frac{i}{h^2}[\psi_2^* - 2\psi_1^* + \psi_0^*] + i\alpha(\psi_2^* + \psi_0^*)\psi_1\psi_1^*, \\
\frac{d}{dt}\psi_2^* &= -\frac{i}{h^2}[\Phi_+^*(\psi_2, \psi_2^*; \psi_1, \psi_1^*; \psi_0, \psi_0^*) - 2\psi_2^* + \psi_1^*] + \\
&+ i\alpha[\Phi_+^*(\psi_2, \psi_2^*; \psi_1, \psi_1^*; \psi_0, \psi_0^*) + \psi_1^*]\psi_2\psi_2^*.
\end{aligned}
\tag{4.8}$$

The next proposition, characterizing the Hamiltonian structure of the reduced dynamical system (4.7) and (4.8), holds.

Proposition 4.1. *The eight-dimensional complex dynamical system (4.7) and (4.8) is Hamiltonian on the manifold $M_h^{(1)} \simeq M_h^8$ with respect to the canonical symplectic structure*

$$\omega_h^{(2)} = \sum_{j=-2,1} (dp_{-j} \wedge d\psi_{-j} + dp_{-j}^* \wedge d\psi_{-j}^*),
\tag{4.9}$$

where, by definition,

$$p_{-j} := \mathcal{L}_{h, \psi_{n-j+1}}^{(1),*}[\psi_n, \psi_n^*] \cdot 1, \quad p_{-j}^* := \mathcal{L}_{h, \psi_{n-j+1}^*}^{(1),*}[\psi_n, \psi_n^*] \cdot 1
\tag{4.10}$$

for $j = \overline{-2, 1}$ modulo the constraint $\text{grad}\mathcal{L}_h^{(1)}[\psi_n, \psi_n^*] = 0, n \in \mathbb{Z}$, on the submanifold $M_h^{(1)} \simeq M_h^8$, and the sign " $\prime, *$ " means the corresponding discrete Frechét up-directed derivative and its natural conjugation with respect to the convolution mapping on $T^*(M_h^{(1)}) \times T(M_h^{(1)})$.

Proof. The symplectic structure (4.9) easily follows [22, 21, 9] from the discrete version of the Gelfand-Dikii [15] differential relationship:

$$(4.11) \quad d\mathcal{L}_h^{(1)}[\psi_n, \psi_n^*] = \langle \text{grad}\mathcal{L}_h^{(1)}[\psi_{n-1}, \psi_{n-1}^*], (d\psi_{n-1}, d\psi_{n-1}^*)^\top \rangle + \frac{d}{dn} \alpha_h^{(1)}(\psi_{n-1}, \psi_{n-1}^*; \psi_n, \psi_n^*; \psi_{n+1}, \psi_{n+1}^*; \psi_{n+2}, \psi_{n+2}^*),$$

where $\alpha_h^{(1)} \in \Lambda^1(M_h^{(1)})$ is, owing to the condition $\text{grad}\mathcal{L}_h^{(1)}[\psi_n, \psi_n^*] = 0, n \in \mathbb{Z}$, on the submanifold $M_h^{(1)}$, not depending on the index $n \in \mathbb{Z}$ and suitably defined one-form on the manifold M_h^8 , allowing the following canonical representation:

$$(4.12) \quad \alpha_h^{(1)} = \sum_{j=\overline{-2, 1}} (p_{-j}(h)d\psi_{-j} + p_{-j}^*(h)d\psi_{-j}^*)$$

with functions $p_{-j}, p_{-j}^* : M_h^{(1)} \times \mathbb{R} \rightarrow \mathbb{C}$, $j = \overline{-2, 1}$. The latter, being defined by the expressions (4.10), compile jointly with variables $\psi_{-j}, \psi_{-j}^* : M_h^{(1)} \times \mathbb{R} \rightarrow \mathbb{C}$, $j = \overline{-2, 1}$, the global coordinates on the finite dimensional symplectic manifold M_h^8 , proving the proposition. \square

The dynamical system (4.7) and (4.8) on the manifold M_h^8 possesses, except its Hamiltonian function, additionally exactly four mutually commuting functionally independent conservation laws $\mathcal{H}_k, \mathcal{H}_k^* : M_h^8 \rightarrow \mathbb{R}$, $k = \overline{0, 1}$, and one Casimir function $\bar{\mathcal{H}}_0 : M_h^8 \rightarrow \mathbb{R}$, which can be calculated [21] from the following functional relationships

$$(4.13) \quad \begin{aligned} & \langle \text{grad}H_k[\psi_n, \psi_n^*], K[\psi_n, \psi_n^*] \rangle := \\ &= -\frac{d}{dn} \mathcal{H}_k(\psi_{n-1}, \psi_{n-1}^*; \psi_n, \psi_n^*; \psi_{n+1}, \psi_{n+1}^*; \psi_{n+2}, \psi_{n+2}^*), \\ & \langle \text{grad}H_k^*[\psi_n, \psi_n^*], K[\psi_n, \psi_n^*] \rangle := \\ &= -\frac{d}{dn} \mathcal{H}_k^*(\psi_{n-1}, \psi_{n-1}^*; \psi_n, \psi_n^*; \psi_{n+1}, \psi_{n+1}^*; \psi_{n+2}, \psi_{n+2}^*), \\ & \langle \text{grad}\bar{H}_0[\psi_n, \psi_n^*], K[\psi_n, \psi_n^*] \rangle := \\ &= -\frac{d}{dn} \bar{\mathcal{H}}_0(\psi_{n-1}, \psi_{n-1}^*; \psi_n, \psi_n^*; \psi_{n+1}, \psi_{n+1}^*; \psi_{n+2}, \psi_{n+2}^*), \end{aligned}$$

for $k = \overline{0, 1}$ modulo the constraint $\text{grad}\mathcal{L}_h^{(1)}[\psi_{n-2}, \psi_{n-2}^*] = 0, n \in \mathbb{Z}$, on the manifold $M_h^{(1)} \simeq M_h^8$, where $\frac{d}{dn} := \Delta - 1$ is a discrete analog of the differentiation and the shift operator Δ acts as $\Delta f_n := f_{n+1}, n \in \mathbb{Z}$, for any mapping $f : \mathbb{Z} \rightarrow \mathbb{C}$. From (4.13) one can obtain by means of simple but tedious calculations analytical expressions for the invariants $\mathcal{H}_k^* : M_h^8 \rightarrow \mathbb{R}$, which give rise to the corresponding Hamiltonian function for the dynamical system (4.7) and (4.8), owing to the relationship (3.5):

$$\mathcal{H} = \frac{2}{\alpha h^3} \bar{\mathcal{H}}_0 + \frac{1}{h} (\mathcal{H}_0 + \mathcal{H}_0^*),$$

satisfying the following canonical Hamiltonian system with respect to the symplectic structure (4.9):

$$(4.14) \quad \begin{aligned} d\psi_{-j}/dt &= \partial\mathcal{H}/\partial p_{-j}, & d\psi_{-j}^*/dt &= \partial\mathcal{H}/\partial p_{-j}^*, \\ dp_{-j}/dt &= -\partial\mathcal{H}/\partial\psi_{-j}, & dp_{-j}^*/dt &= -\partial\mathcal{H}/\partial\psi_{-j}^*, \end{aligned}$$

where $j = \overline{-2, 1}$, which is a Liouville-Arnold integrable on the symplectic manifold M_h^8 .

Remark 4.2. The same way on can construct the finite dimensional reduction of the discrete Schrödinger dynamical system (1.7) in the case $N = 2$. Making use of the calculated before conservation laws (3.3), one can take the corresponding action functional as

$$\begin{aligned}
(4.15) \quad \mathcal{L}_h^{(2)} &= \bar{c}_0(h) \sum_{n \in \mathbb{Z}} \ln |1 - \alpha h^2 \psi_n^* \psi_n| + c_0(h) \sum_{n \in \mathbb{Z}} (\psi_n^* \psi_{n-1} + \psi_n \psi_{n-1}^*) + \\
&+ c_1(h) \sum_{n \in \mathbb{Z}} \left(\frac{1}{2} \psi_n^2 \psi_{n-1}^{*,2} + \psi_n \psi_n^* (\psi_{n+1} \psi_{n-1}^* + \psi_{n-1} \psi_{n+1}^*) \right. \\
&+ \left. \frac{1}{2} \psi_{n-1}^2 \psi_n^{*,2} - \frac{\psi_n \psi_{n-2}^*}{\alpha h^2} - \frac{\psi_{n-2} \psi_n^*}{\alpha h^2} \right) + \\
&+ c_2(h) \sum_{n \in \mathbb{Z}} \left(\frac{1}{3} \psi_n^3 \psi_{n-1}^{*,3} + \psi_n \psi_{n+1} \psi_{n-1}^* \psi_n^* (\psi_n \psi_{n-1}^* + \psi_{n+1} \psi_n^* + \right. \\
&+ \left. \psi_{n+2} \psi_{n+1}^*) - \frac{\psi_n \psi_{n-1}^*}{\alpha h^2} (\psi_n \psi_{n-2}^* + \psi_{n+1} \psi_{n-1}^*) - \right. \\
&- \left. \frac{\psi_n \psi_n^*}{\alpha h^2} (\psi_{n+1} \psi_{n-2}^* + \psi_{n+2} \psi_{n-1}^*) + \frac{\psi_n \psi_{n-3}^*}{\alpha^2 h^4} + \frac{1}{3} \psi_n^{*,3} \psi_{n-1}^3 + \right. \\
&+ \left. \psi_n^* \psi_{n+1}^* \psi_{n-1} \psi_n (\psi_n^* \psi_{n-1} + \psi_{n+1}^* \psi_n + \psi_{n+2}^* \psi_{n+1}) - \right. \\
&- \left. \frac{\psi_n^* \psi_{n-1}}{\alpha h^2} (\psi_n^* \psi_{n-2} + \psi_{n+1}^* \psi_{n-1}) - \frac{\psi_n \psi_n^*}{\alpha h^2} (\psi_{n+1}^* \psi_{n-2} + \psi_{n+2}^* \psi_{n-1}) + \frac{\psi_n^* \psi_{n-3}}{\alpha^2 h^4} \right)
\end{aligned}$$

with mappings $\bar{c}_0, c_j : \mathbb{R}_+ \rightarrow \mathbb{R}$, $j = \overline{0, 2}$, defined as before from the condition that there exists the limit

$$(4.16) \quad \tilde{\mathcal{L}}^{(2)} := \lim_{h \rightarrow 0} \mathcal{L}_h^{(2)}.$$

The respectively defined critical submanifold

$$(4.17) \quad M_h^{(2)} := \left\{ (\psi, \psi^*)^\top \in M_h : \text{grad} \mathcal{L}_h^{(2)}[\psi_n, \psi_n^*] = 0, n \in \mathbb{Z} \right\}$$

becomes diffeomorphic to a finite dimensional canonically symplectic manifold M_h^{12} on which the suitably reduced discrete Schrödinger dynamical system (1.7) becomes a Liouville-Arnold integrable Hamiltonian system. The details of the related calculations are planned to be presented in a separate work under preparation.

5. THE FOURIER ANALYSIS OF THE INTEGRABLE DISCRETE NLS SYSTEM

It easy to observe that the linearized Schrödinger system (1.1) admits the following Fourier type solution:

$$(5.1) \quad \psi(x, t) = \int_{\mathbb{R}} ds \xi(s, t) \exp(ixs), \quad \psi^*(x, t) = \int_{\mathbb{R}} ds \xi^*(s, t) \exp(-ixs)$$

for all $x, t \in \mathbb{R}$, where $d\xi/dt = -is^2\xi$, $d\xi^*/dt = is^2\xi^*$, i.e.,

$$(5.2) \quad \xi(s, t) = \bar{\xi}(s) e^{-is^2 t}, \quad \xi^*(s, t) = \bar{\xi}^*(s) e^{is^2 t}$$

and $\bar{\xi}, \bar{\xi}^* : \mathbb{R} \rightarrow \mathbb{C}$ are prescribed functions (the Fourier transforms of initial conditions). Likewise, the linearized discrete Schrödinger dynamical system (1.7) allows the following general discrete Fourier type solution:

$$(5.3) \quad \psi_n = \int_{\mathbb{R}} ds \xi_h(s, t) \exp(ihns), \quad \psi_n^* = \int_{\mathbb{R}} ds \xi_h^*(s, t) \exp(-ihns)$$

for all $n \in \mathbb{Z}$, where the evolution parameter $t \in \mathbb{R}$, $(\psi_n, \psi_n^*)^\top \in w_{h,2}^2(\mathbb{Z}; \mathbb{C}^2)$ and

$$(5.4) \quad \xi_h(s, t) = \bar{\xi}_h(s) \exp(-i \frac{4t}{h^2} \sin^2 \frac{sh}{2}), \quad \xi_h^*(s, t) = \bar{\xi}_h^*(s) \exp(i \frac{4t}{h^2} \sin^2 \frac{sh}{2}).$$

Here the function $(\bar{\xi}_h, \bar{\xi}_h^*)^\top \in W_{h,2}^2(\mathbb{R}; \mathbb{C}^2) \subset L_2(\mathbb{R}; \mathbb{C}^2)$, where the functional space $W_{h,2}^2(\mathbb{R}; \mathbb{C}^2)$ is yet to be determined. From the boundary condition $(\psi_n, \psi_n^*)^\top \in w_{h,2}^2(\mathbb{Z}; \mathbb{C}^2)$ it follows that

expressions

$$(5.5) \quad \begin{aligned} \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \psi_n^* \psi_n &= \int_{\mathbb{R}} ds \xi_h^*(s) \xi_h(s) < \infty, \\ \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} (\psi_{n+1}^* \psi_n + \psi_n^* \psi_{n+1}) &= 2 \int_{\mathbb{R}} ds \cos(hs) \xi_h^*(s) \xi_h(s) < \infty, \end{aligned}$$

ensure the boundedness of the Hamiltonian function (1.11), thereby determining a functional space $W_{h,2}^2(\mathbb{R}; \mathbb{C}^2)$ to which belong the vector function $(\xi_h, \xi_h^*)^\top \in L_2(\mathbb{R}; \mathbb{C}^2)$. However the discrete evolution is not following along the continuous trajectory.

Being motivated by works [11, 13], we modify the discrete system as follows in order to obtain the exact discretization:

$$(5.6) \quad \begin{aligned} \frac{d}{dt} \psi_n &= \frac{i}{\delta^2} (\psi_{n+1} - 2\psi_n + \psi_{n-1}) - i\alpha(\psi_{n+1} + \psi_{n-1}) \psi_n \psi_n^*, \\ \frac{d}{dt} \psi_n^* &= -\frac{i}{\delta^2} (\psi_{n+1}^* - 2\psi_n^* + \psi_{n-1}^*) + i\alpha(\psi_{n+1}^* + \psi_{n-1}^*) \psi_n \psi_n^*, \end{aligned}$$

Substituting (5.3) into the linearization of (5.6) we obtain

$$(5.7) \quad \xi_h(s, t) = \bar{\xi}_h(s) \exp(-i \frac{4t}{\delta^2} \sin^2 \frac{sh}{2}), \quad \xi_h^*(s, t) = \bar{\xi}_h^*(s) \exp(i \frac{4t}{\delta^2} \sin^2 \frac{sh}{2}).$$

Therefore, linearization of the discretization (5.6) is exact (i.e., $\psi(nh, t) = \psi_n(t)$, $n \in \mathbb{Z}$, if we assume

$$(5.8) \quad \delta = \frac{2}{s} \sin \frac{hs}{2}$$

for any $h \in \mathbb{R}$. Thus, the parameter $\delta > 0$ depends on $s \in \mathbb{R}$ yet for small $h \rightarrow 0$ one gets $\delta = h(1 + O(h^2 s^2))$.

The nonlinear case is more difficult. In the continuous nonlinear case (5.1) we have

$$(5.9) \quad d\xi/dt = -is^2 \xi - 2i\alpha\beta[s; \xi], \quad d\xi^*/dt = is^2 \xi^* + 2i\alpha\beta^*[s; \xi^*],$$

where the functionals $\beta, \beta^* : \mathbb{R} \times L_2(\mathbb{R}; \mathbb{C}) \rightarrow L_2(\mathbb{R}; \mathbb{C})$, determined as

$$\begin{aligned} \beta[s; \xi] &: = \int_{\mathbb{R}^2} ds' ds'' \xi(s + s' - s'') \xi(s'') \xi^*(s'), \\ \beta^*[s; \xi] &: = \int_{\mathbb{R}^2} ds' ds'' \xi^*(s + s' - s'') \xi^*(s'') \xi(s'), \end{aligned}$$

depend both on $s \in \mathbb{R}$ and on the element $\xi \in L_2(\mathbb{R}; \mathbb{C})$, as well as depends parametrically on the evolution parameter $t \in \mathbb{R}$ through (5.9). In the nonlinear discrete case (5.6) we, respectively, obtain:

$$(5.10) \quad d\xi_h/dt = -i\xi_h \frac{4}{\delta^2} \sin^2 \frac{sh}{2} - 2i\alpha\beta_h[s; \xi_h], \quad d\xi_h^*/dt = i\xi_h^* \frac{4}{\delta^2} \sin^2 \frac{sh}{2} + 2i\alpha\beta_h^*[s; \xi_h],$$

where the functionals $\beta_h, \beta_h^* : \mathbb{R} \times L_2(\mathbb{R}; \mathbb{C}) \rightarrow L_2(\mathbb{R}; \mathbb{C})$ are determined as

$$(5.11) \quad \begin{aligned} \beta_h[s; \xi_h] &: = \int_{\mathbb{R}^2} ds' ds'' \cos[h(s + s' - s'')] \xi_h(s + s' - s'') \xi_h(s'') \xi_h^*(s'), \\ \beta_h^*[s; \xi_h] &: = \int_{\mathbb{R}^2} ds' ds'' \cos[h(s + s' - s'')] \xi_h^*(s + s' - s'') \xi_h^*(s'') \xi_h(s') \end{aligned}$$

for any $s \in \mathbb{R}$. To proceed further with the truly nonlinear case still persists to be a nontrivial problem, yet we hope to obtain a suitable procedure analogous to that of [12, 14].

Instead of it one can analyze the related functional space constraints on the space of functions $(\bar{\xi}_h, \bar{\xi}_h^*)^\top \in W_{h,2}^2(\mathbb{R}; \mathbb{C}^2)$, representing solutions to the discrete nonlinear equation (1.7) via the expressions (5.3), being imposed by the corresponding finite dimensional reduction scheme of Section (4). This procedure actually may be realized, if to consider the derived before recurrence relationships (4.5) (or similarly, (4.6)) allowing to obtain the related constraints on the space of

functions $(\bar{\xi}_h, \bar{\xi}_h^*)^\top \in W_{h,2}^2(\mathbb{R}; \mathbb{C}^2)$, but the resulting relationships prove to be much complicated and cumbersome expressions.

Thus, one can suggest the following practical numerical-analytical scheme of constructing solutions to the discrete nonlinear Schrödinger dynamical system (1.7): first to solve the Cauchy problem to the finite-dimensional system of ordinary differential equations (4.7) and (4.8), and next to substitute them into the system of recurrent algebraic relationships (4.5) and (4.6), obtaining this way the whole infinite hierarchy of the sought for solutions.

6. CONCLUSION

Within the presented investigation of solutions to the discrete nonlinear Schrödinger dynamical system (1.7) we have succeeded in two important points. First, we have developed an effective enough scheme of invariant reducing the infinite system of ordinary differential equations (1.7) to an equivalent finite one of ordinary differential equations with respect to the evolution parameter $t \in \mathbb{R}$. Second, we constructed a finite set of recurrent algebraic regular relationships, allowing to expand the obtained before solutions to any discrete order $n \in \mathbb{Z}$ and giving rise to the sought for solutions of the system (1.7).

It is important to mention here that within the presented analysis we have not used the Lax type representation for the discrete nonlinear Schrödinger dynamical system (1.7), whose existence was stated many years ago in [1] and whose complete solution set analysis was done in works [1, 2, 10, 20] by means of both the inverse scattering transform and the algebraic-geometric methods. Concerning the set of recurrent relationships for exact solutions to the finite-dimensional reduction of the discrete nonlinear Schrödinger dynamical system (1.7), obtained both in the presented work and in work [10], based on the corresponding Lax type representation, an interesting problem of finding between them relationship arises, and an answer to it would explain the hidden structure of the complete Liouville-Arnold integrability of the related set of the reduced ordinary differential equations.

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